Quiz 5 Solutions

1. Decide with full justification whether the series converges:

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}.$$

Solution: Since for large n we have $\sqrt{n^2-1} \approx \sqrt{n^2} = n$, we compare with $\dfrac{1}{n^2}$. Then

$$\frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \frac{n^2}{n\sqrt{n^2-1}} = \frac{1}{\sqrt{1-\frac{1}{n^2}}}.$$

This has limit 1 as $n \to \infty$. We know that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$), hence by the limit comparison test $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ also converges.

The integral test also works: $f(x) = \dfrac{1}{x\sqrt{x^2-1}}$ is decreasing, continuous and goes to 0 as $x \to \infty$. We have

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \operatorname{arcsec} x \Big|_2^t \\
&= \lim_{t\to\infty} \operatorname{arcsec}(t) - \operatorname{arcsec}(2) \\
&= \frac{\pi}{2} - \frac{\pi}{3}.
\end{aligned}$$

So by the integral test, the series converges. (To see $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \operatorname{arcsec} x + C$, do trig sub $x = \sec\theta$)

2. Decide with full justification whether the series converges absolutely, conditionally, or diverges:

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}.$$

Solution: This is an alternating series with $b_n = \dfrac{1}{\sqrt{\ln n}}$. Since $\sqrt{\ }$ and ln are increasing, b_n is decreasing. Also $b_n \to 0$ as $n \to \infty$. So the alternating series test is applicable and shows that the series converges. So we only have to decide whether it converges absolutely or conditionally. We have $\left|\dfrac{(-1)^n}{\sqrt{\ln n}}\right| = \dfrac{1}{\sqrt{\ln n}} > \dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (Harmonic series). So by the comparison test, $\displaystyle\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{\sqrt{\ln n}}\right|$ diverges, hence the series in question is conditionally convergent.

3. Decide with full justification whether the series converges absolutely, conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{n(-3)^n}{n!}.$$

Solution: Let $a_n = \dfrac{n(-3)^n}{n!}$. We apply the ratio test:

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)(-3)^{n+1}}{(n+1)!} \frac{n!}{n(-3)^n}\right| \\
&= \frac{n+1}{n} \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!} \\
&= \frac{n+1}{n} 3 \frac{1}{n+1} \\
&= \frac{3}{n}
\end{aligned}$$

This goes to $0 < 1$ as $n \to \infty$, therefore the series converges absolutely by the ratio test.